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LETTER TO THE EDITOR

Symmetry of the Calogero model confined in the harmonic potential—Yangian and W algebra

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Abstract. The $su(\nu)$ Calogero model confined in the harmonic potential is considered. We show that the model has Yangian symmetry $Y(su(\nu))$. Unifying with the creation–annihilation operators which constitute the $su(\nu)$ loop algebra, we reveal the algebraic structure of this system. The relation with the energy spectrum is also discussed.

Yangian symmetry has received much interest in a recent development of mathematical physics. The Yangian was first introduced by Drinfeld [1] as the Hopf algebra of a quantum group (for a review see [2, 3]). This symmetry is accompanied by Yang's R-matrix, which is a rational solution of the quantum Yang–Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)\,. \tag{1}$$

The Yangian $Y(su(\nu))$ are defined by taking the two lowest generators Q_0^{ab} and Q_1^{ab}, which satisfy the following relations:

$$\left[Q_0^{ab}, Q_0^{cd}\right] = \delta^{bc} Q_0^{ad} - \delta^{da} Q_0^{cb} \tag{2a}$$

$$\left[Q_0^{ab}, Q_1^{cd}\right] = \delta^{bc} Q_1^{ad} - \delta^{da} Q_1^{cb} \tag{2b}$$

$$\begin{aligned}\left[Q_0^{ab}, [Q_1^{cd}, Q_1^{ef}]\right] &- [Q_1^{ab}, [Q_0^{cd}, Q_1^{ef}]] \\ &= \tfrac{1}{4}\alpha^2 \left(\left[Q_0^{ab}, \left[(Q_0Q_0)^{cd}, (Q_0Q_0)^{ef}\right]\right] - \left[(Q_0Q_0)^{ab}, \left[Q_0^{cd}, (Q_0Q_0)^{ef}\right]\right]\right)\end{aligned} \tag{2c}$$

in which, for the sake of brevity, we denote $(Q_0Q_0)^{ab}$ as $\sum_{p=1}^{\nu} Q_0^{ap} Q_0^{pb}$. The third equation (2$c$) is called the 'deformed' Serré relation. This relation indicates that the Yangian $Y(su(\nu))$ is not the Lie algebra, but a generalization of the Lie algebra with the 'deformation' parameter α. Note that in the limit $\alpha \to 0$ the Yangian $Y(su(\nu))$ reduces to $su(\nu)$ loop algebra. The Hopf co-multiplication is given by

$$\Delta Q_0^{ab} = Q_0^{ab} \otimes 1 + 1 \otimes Q_0^{ab}$$

$$\Delta Q_1^{ab} = Q_1^{ab} \otimes 1 + 1 \otimes Q_1^{ab} - \tfrac{1}{2}\alpha \sum_{c=1}^{\nu} \left(Q_0^{ac} \otimes Q_0^{cb} - Q_0^{cb} \otimes Q_0^{ac}\right).$$

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Recently the Yangian $Y(su(\nu))$ has been used in the one-dimensional quantum $su(\nu)$ spin system with inverse-square interactions. There are two types of the integrable and dynamical N-body spin systems [4–6], whose Hamiltonians are given by

$$\mathcal{H}_{\rm C} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \frac{\lambda^2 - \lambda P_{jk}}{(x_j - x_k)^2} \tag{3}$$

$$\mathcal{H}_{\rm S} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \frac{\lambda^2 - \lambda P_{jk}}{\sinh^2(z_j - z_k)} . \tag{4}$$

We call (3) and (4) the Calogero-type and the Sutherland-type model, respectively. Here and hereafter we use $X^{ab} = |a\rangle\langle b|$ (for $a, b = 1, \ldots, \nu$) as a basis of the $su(\nu)$ spin operators. In terms of this basis we can denote the permutation operator P_{jk} by

$$P_{jk} = \sum_{a,b=1}^{\nu} X_j^{ab} X_k^{ba} .$$

The symmetries of these two dynamical spin systems are different. While the Calogero-type (3) constitutes the $su(\nu)$ loop algebra [7], the Sutherland-type (4) has the Yangian $Y(su(\nu))$ structure [8–10].

In this letter we consider the $su(\nu)$ Calogero model confined in the external harmonic potential (the confined $su(\nu)$ Calogero model, for short). The Hamiltonian is written as

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \frac{\lambda^2 - \lambda P_{jk}}{(x_j - x_k)^2} + \omega^2 \sum_{j=1}^{N} x_j^2 . \tag{5}$$

First we shall show that this model remains to be integrable. The integrability can be shown based on the Lax formalism. We set $N \times N$ operator-valued matrices as follows:

$$\mathbf{L}_{jk} = \delta_{jk} \frac{\partial}{\partial x_j} - (1 - \delta_{jk}) \frac{\lambda P_{jk}}{x_j - x_k}$$

$$\mathbf{M}_{jk} = \delta_{jk} \sum_{s\neq j}^{N} \frac{2\lambda P_{js}}{(x_j - x_s)^2} - (1 - \delta_{jk}) \frac{2\lambda P_{jk}}{(x_j - x_k)^2}$$

$$\mathbf{Q}_{jk} = \delta_{jk} x_j .$$

In terms of these matrices, $\mathbf{L}$ and $\mathbf{Q}$, we define for our convenience $\mathbf{L}^{\pm} = \mathbf{L} \mp \omega \mathbf{Q}$. Matrices $\mathbf{L}^{\pm}$ satisfy the quantum Lax equation

$$\begin{aligned}[\mathcal{H}, \mathbf{L}_{jk}^{\pm}] &= \sum_{l=1}^{N} \left(\mathbf{L}_{jl}^{\pm} \mathbf{M}_{lk} - \mathbf{M}_{jl} \mathbf{L}_{lk}^{\pm}\right) \pm 2\omega \mathbf{L}_{jk}^{\pm} \\ &= [\mathbf{L}^{\pm}, \mathbf{M}]_{jk} \pm 2\omega \mathbf{L}_{jk}^{\pm} .\end{aligned} \tag{6}$$

Using the Lax equation (6) and the 'sum-to-zero' condition, $\sum_k \mathbf{M}_{jk} = \sum_j \mathbf{M}_{jk} = 0$, one can find that the operators defined by

$$T_n^{ab} = \sum_{j,k} X_j^{ab} \left((\mathbf{L}^{+}\mathbf{L}^{-})^n\right)_{jk} \tag{7}$$

commute with the Hamiltonian, $[\mathcal{H}, T_n^{ab}] = 0$. This fact proves the quantum integrability of the confined $su(\nu)$ Calogero model (5) in the Liouville sense. Notice that the wavefunctions can be derived explicitly in the forms of integral representation [11, 12].

We shall show that the confined $su(\nu)$ Calogero model (5) has the Yangian symmetry as in the case of the Sutherland-type model (4). To begin with, we define several generators as follows:

$$J_0^{ab} = \sum_j X_j^{ab} \tag{8}$$

$$J_1^{ab} = \sum_j X_j^{ab}\frac{\partial}{\partial x_j} - \lambda \sum_{j,k}{}' (X_j X_k)^{ab}\frac{1}{x_j - x_k} \tag{9}$$

$$J_2^{ab} = \sum_j X_j^{ab}\frac{\partial^2}{\partial x_j^2} - \lambda \sum_{j,k}{}' (X_j X_k)^{ab}\left(\frac{1}{x_j - x_k}\left(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k}\right) - \frac{1}{(x_j - x_k)^2}\right)$$
$$-\lambda^2 \sum_{j,k}{}' X_j^{ab}\frac{1}{(x_j - x_k)^2} + \lambda^2 \sum_{j,k,l}{}' (X_j X_k X_l)^{ab}\frac{1}{(x_j - x_k)(x_k - x_l)}\,. \tag{10}$$

Here we have used notation, $(X_j X_k)^{ab} = \sum_{p=1}^{\nu} X_j^{ap} X_k^{pb}$, etc. The symbol $\sum'$ means any two summation indices does not coincide. The generators J_n^{ab} are the conserved operators of the Calogero-type (3), and generally defined by $J_n^{ab} = \sum_{j,k} X_j^{ab}(\mathbf{L}^n)_{jk}$. These generators satisfy the commutation relation of the loop algebra [7]

$$[J_n^{ab}, J_m^{cd}] = \delta^{bc} J_{n+m}^{ad} - \delta^{da} J_{n+m}^{cb}\,. \tag{11}$$

Further we have another simple representation of the $su(\nu)$ loop algebra. One sees that generators K_n^{ab}, defined by

$$K_n^{ab} = \sum_j X_j^{ab} x_j^n \tag{12}$$

also constitute the loop algebra,

$$[K_n^{ab}, K_m^{cd}] = \delta^{bc} K_{n+m}^{ad} - \delta^{da} K_{n+m}^{cb}\,. \tag{13}$$

Based on our generators $\{J_n^{ab}, K_n^{ab}\}$ we can see the Yangian symmetry of the Sutherland-type [9]. It is interesting that one has another representation of the Yangian $Y(su(\nu))$ from these generators $\{J_n^{ab}, K_n^{ab}\}$. After lengthy calculation we find that generators Q_0^{ab} and Q_1^{ab} are defined by

$$Q_0^{ab} = J_0^{ab} \tag{14}$$
$$Q_1^{ab} = J_2^{ab} - \omega^2 K_2^{ab} \tag{15}$$

satisfy the Yangian's relations (2) with $\alpha^2 \equiv 4\lambda^2\omega^2$. As the generators Q_0^{ab} and Q_1^{ab} are the conserved operators of $\mathcal{H}$, $[\mathcal{H}, Q_0^{ab}] = [\mathcal{H}, Q_1^{ab}] = 0$, one can conclude that the confined $su(\nu)$ Calogero model has the Yangian symmetry as in the case of the Sutherland-type (4). In cases of $\lambda = 0$ ('coloured' harmonic oscillator) and $\omega = 0$ (Calogero-type model) the Yangian symmetry $Y(su(\nu))$ reduces to the $su(\nu)$ loop algebra.

The Yangian symmetry for the confined $su(\nu)$ Calogero model can be shown directly by constructing the transfer matrix $\mathbf{T}(u)$. For this purpose, we introduce operators [13, 6],

$$D_j^{\pm} = \frac{\partial}{\partial x_j} - \sum_{k \neq j} \frac{\lambda}{x_j - x_k} K_{jk} \mp \omega x_j \tag{16}$$

where operator K_{jk} exchanges the positions x_j; $K_{jk}x_j = x_k K_{jk}$. These operators $D_j^{\pm}$ are called the Dunkl operators in mathematical literature. The Dunkl operators satisfy the following relations:

$$[D_j^\pm, D_k^\pm] = 0 \tag{17a}$$

$$K_{jk} D_j^\pm = D_k^\pm K_{jk} \tag{17b}$$

$$[D_j^-, D_j^+] = -2\omega\Big(1 + \lambda \sum_{l\neq j} K_{jl}\Big) \tag{17c}$$

$$[D_j^-, D_k^+] = 2\lambda\omega K_{jk} \qquad \text{for} \quad j \neq k\,. \tag{17d}$$

From these relations we have the identity,

$$(u - v + 2\lambda\omega K_{jk}) \frac{1}{v - D_k^+ D_k^-} \frac{1}{u - D_j^+ D_j^-} = \frac{1}{u - D_j^+ D_j^-} \frac{1}{v - D_k^+ D_k^-} (u - v + 2\lambda\omega K_{jk})\,. \tag{18}$$

Following the method of [9] we define the projection π, which replace the exchange operator K_{jk} by permutation operator P_{jk} after it has been moved to the right of an expression. Then we have the transfer matrix

$$\begin{aligned} \mathbf{T}_0(u) &= 1 + 2\lambda\omega \sum_{j,k=1}^{N} \mathbf{P}_{0j} \left(\frac{1}{u - \mathbf{L}^+\mathbf{L}^-}\right)_{jk} \\ &= \pi\left(1 + 2\lambda\omega \sum_{j=1}^{N} \frac{\mathbf{P}_{0j}}{u - D_j^+ D_j^-}\right). \end{aligned} \tag{19}$$

Here we denote the auxiliary space as 0. By using identity (18) we can obtain the Yang–Baxter equation for the transfer matrix $\mathbf{T}(u)$,

$$\mathbf{R}_{00'}(u - v)\mathbf{T}_0(u)\mathbf{T}_{0'}(v) = \mathbf{T}_{0'}(v)\mathbf{T}_0(u)\mathbf{R}_{00'}(u - v)\,. \tag{20}$$

Here matrix $\mathbf{R}$ is the rational solution of the Yang–Baxter equation (1)

$$\mathbf{R}_{00'}(u) = u + 2\lambda\omega P_{00'}\,. \tag{21}$$

By the definition of Drinfeld [1] we can conclude, in fact, that the transfer matrix $\mathbf{T}(u)$ of the confined $su(\nu)$ Calogero model has a structure of Yangian $Y(su(\nu))$. When we expand $\mathbf{T}(u)$ in u as $\mathbf{T}(u) = \sum_{k\geqslant 0} u^{-k-1}\mathbf{T}_k$, generators $\{Q_0^{ab}, Q_1^{ab}\}$ in (14) and (15) can be obtained from $\{\mathbf{T}_0, \mathbf{T}_1\}$.

Besides the trivial conserved operators defined by $\sum_{j,k}((\mathbf{L}^+\mathbf{L}^-)^n)_{jk}$, we have the generating function of the conserved operators. The bilinear relation (20) shows that the generating function of the Yangian-invariant operators is given by a quantum determinant of $\mathbf{T}(u)$,

$$\mathrm{qDet}\,\mathbf{T}(u) \equiv \sum_{\sigma} (-)^\sigma T^{1\sigma(1)}(u) T^{2\sigma(2)}(u + 2\lambda\omega) \ldots T^{\nu\sigma(\nu)}(u + 2(n-1)\lambda\omega)\,. \tag{22}$$

This quantum determinant is indeed a centre of Yangian and commutes with all elements of $\mathbf{T}(u)$,

$$[\mathrm{qDet}\,\mathbf{T}(v), \mathbf{T}(u)] = 0\,. \tag{23}$$

We have seen that the conserved operators of the confined $su(\nu)$ Calogero model have the Yangian symmetry. We stress that not only the Yangian $Y(su(\nu))$ but the original $su(\nu)$ loop algebras are included in this model. We introduce the creation–annihilation operators of the Hamiltonian $\mathcal{H}$. We denote them as $\mathbf{B}_{n\pm}$, and set

$$B_{0\pm}^{ab} = Q_0^{ab} \tag{24}$$

$$B_{1\pm}^{ab} = J_1^{ab} \mp \omega K_1^{ab}\,. \tag{25}$$

One finds that two sets of generators $\mathbf{B}_{n\pm}$, which are defined by $B^{ab}_{n\pm} = \sum_{j,k} X^{ab}_j((\mathbf{L}^{\pm})^n)_{jk}$, constitute the $su(\nu)$ loop algebra,

$$\left[B^{ab}_{n\pm}, B^{cd}_{m\pm}\right] = \delta^{bc} B^{ad}_{n+m\pm} - \delta^{da} B^{cb}_{n+m\pm} . \tag{26}$$

The meaning of the generators $\mathbf{B}_{n\pm}$ will be clear from the relation

$$[\mathcal{H}, B^{ab}_{n\pm}] = \pm 2n\omega B^{ab}_{n\pm} . \tag{27}$$

This commutation relation shows that generators $\mathbf{B}_{n\pm}$ are the creation–annihilation operators for the Hamiltonian of the confined Calogero model.

It has been revealed that the confined $su(\nu)$ Calogero model realizes algebra including both the Yangian and two sets of the loop algebra as subalgebra. We should note that the Yangian and the loop algebra are not independent of each other. This can be checked from the fact that the Yangian's generator Q^{ab}_1 can be defined by the use of $\mathbf{B}_{1\pm}$:

$$\left[B^{ab}_{1+}, B^{cd}_{1-}\right] + \left[B^{ab}_{1-}, B^{cd}_{1+}\right] = 2\left(\delta^{bc} Q^{ad}_1 - \delta^{da} Q^{cb}_1\right) . \tag{28}$$

We also have the 'Serré-like' relation,

$$\left[Q^{ab}_0, \left[B^{cd}_{1\pm} + B^{cd}_{2\mp}, B^{ef}_{1\pm} + B^{ef}_{2\mp}\right]\right] - \left[B^{ab}_{1\pm} + B^{ab}_{2\mp}, \left[Q^{cd}_0, B^{ef}_{1\pm} + B^{ef}_{2\mp}\right]\right] = 0 . \tag{29}$$

Under the condition of (2), (26), (28) and (29), we can consistently define algebra constructed from three generators, $\{\mathbf{B}_0, \mathbf{B}_{1\pm}\}$. This algebra includes two sets of loop algebra $\{\mathbf{B}_{n\pm}\}$, and Yangian $\{\mathbf{Q}_n\}$. The situation of these generators is drawn in figure 1. In this sense the algebra constructed here might be regarded as W-algebra with 'deformed' Lie algebra which has Hopf structure [14, 15]. Notice that the $U(1)$ current realizes the W_∞ algebra in the limit $N \to \infty$ [16].

The energy spectrum of the confined $su(\nu)$ Calogero model (5) can be obtained by using creation–annihilation operators $\mathbf{B}_{n\pm}$ [6, 11]. Due to these generators the energy spectrum has a simple form,

$$E - E_0 = \lambda\omega C_2 + 2\omega \sum_{k>0} k N_k . \tag{30}$$

Here C_2 is an eigenvalue of the second Casimir operator $\sum_{a=1}^{\nu} (J_0 J_0)^{aa}$, and N_k are non-negative integers. From the point of the Yangian symmetry (20) the confined $su(\nu)$ Calogero model is classified by deformation parameter $\lambda \times \omega$, but its energy spectrum depends not only on $\lambda \times \omega$ but on ω. This fact will help us to understand the relationship between the Yangian symmetry $Y(su(\nu))$ and the Wess–Zumino–Witten theory [17, 18].

We have established the structure of the confined $su(\nu)$ Calogero model. We have constructed algebra, which includes the Yangian $Y(su(\nu))$ and $su(\nu)$ loop algebras as subalgebras. When we change a basis of algebra from $\{\mathbf{B}_0, \mathbf{B}_{1\pm}\}$ to $\{\mathbf{J}_0, \mathbf{J}_1, \mathbf{K}_1\}$, one can combine the Calogero-type (3) and the Sutherland-type (4) [7].

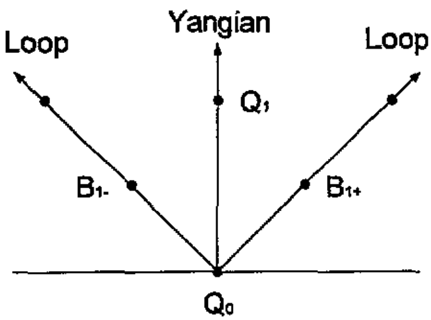

Figure 1. The generators Q^{ab}_n and $B^{ab}_{n\pm}$ have been plotted. The x- and y-axes represent Laurent mode and conformal spin of generators, respectively.

It is also known that the lattice integrable spin system of the Haldane–Shastry type [19, 20] can be obtained from the confined $su(\nu)$ Calogero model discussed in this letter. Such lattice spin model, which may be called the Polychronakos–Frahm spin model, is not translationally invariant, but preserves Yangian symmetry. As in the case of the confined Calogero model the Hamiltonian of Polychronakos–Frahm spin model can be identified with the Virasoro generator L_0 as well, and its partition function in large-N limit coincides with the Virasoro character formula for level-1 WZW theory [21]. This agrees with the result of [17] that the Yangian invariant system can be embedded in the WZW theory, and that the Virasoro generator L_0 is the first conservative of Yangian currents.

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